4 Hydrodynamics of stratified waters

4.1 Navier-Stokes equation in a rotating reference frame

The following equations are derived from the basic principles of conservation of mass, momentum, and energy. Sometimes it is necessary to consider a finite arbitrary volume, called a control volume, over which these principles can be applied. The control volume can remain fixed in space (Eulerian view) or can move with the fluid (Lagrangian view).

a) Material and substantive (or total) derivative

Changes of the currents of a moving environmental fluid can be measured in two different ways:

> (i) We can measure its changes with the help of an anemometer, such as by a weather station (atmosphere) or by a moored current meter (water).
>
> (ii) The alternative is the measurement on a floating platform such as a weather balloon (atmosphere) or drifting / floating buoy (water).

The difference between these two observations is obvious: The current meter in (i) is measuring the velocity of all the moving particles passing by a **fixed point in space**, whereas in (ii) the instrument is measuring changes in velocity as it **moves with the fluid**. The same situation arises in measuring changes in density, temperature, salinity, etc. Therefore, when formulating the changes (dC/dt), we have to separate out these two different views. The derivative (d ../dt) of a field with respect to fixed positions in space is called the **spatial** or **Eulerian derivative**. The derivative following a moving particle is called the **substantive**, **Lagrangian** or **material derivative**.

The substantive derivative is the total derivative in time and defined as the operator:

(4.1.1) $$\frac{D}{Dt}(*) \equiv \frac{\partial}{\partial t}(*) + \left(\vec{u} \cdot \nabla\right)(*)$$

where $\vec{u}$ is the velocity (3-dimensional) vector of the flow. The first term on the right-hand side of the equation is the ordinary **Eulerian derivative** (i.e. the derivative in a fixed reference frame), whereas the second term represents the changes brought about by the moving fluid. This effect is referred to as **advection**. Mathematically: $\left(\vec{u} \cdot \nabla\right)$ is the vector product between the velocity $\vec{u}$ and the **Nabla-operator** $\nabla = (\partial/\partial x,\ \partial/\partial y, \partial/\partial z)$.

b) Conservation of mass - Continuity Equation

The conservation of mass is expressed by

(4.1.2) $$\frac{\partial \rho}{\partial t} + \sum_j \frac{\partial}{\partial x_j}(\rho u_j) = 0$$ or in vector form $$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{u}) = 0 .$$

Equivalent is (by applying the calculus rules for derivatives of products):

(4.1.3) $$\frac{\partial \rho}{\partial t} + \sum_j u_j \frac{\partial \rho}{\partial x_j} + \rho \sum_j \frac{\partial u_j}{\partial x_j} = 0$$ or $$\frac{\partial \rho}{\partial t} + \vec{u} \cdot \operatorname{grad}(\rho) + \rho \operatorname{div}(\vec{u}) = 0$$

The sum of the first two terms $\frac{\partial\rho}{\partial t}+\sum_j u_j \frac{\partial\rho}{\partial x_j}$ corresponds exactly to the definition of the substantive (or total) derivative $\frac{D\rho}{Dt}$ (see definition above). If $\frac{1}{\rho}\frac{D\rho}{Dt} << \frac{\partial u_j}{\partial x_j}$, then the relative change of the mass in a control volume is completely determined by the flow pattern (and not by density changes), and we can conclude:

(4.1.4) $$\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}=0 \qquad \text{or} \qquad \mathrm{div}(\vec{u})=0 \qquad \text{or} \qquad \sum_j \frac{\partial u_j}{\partial x_j}=0$$

This approximation is well fulfilled in natural waters, as relative density changes Δρ/ρ can be ignored in comparison to relative current changes (Δρ/ρ < ~3x10^{-3} over typical depth ranges of a few meters during stratification and for seasonal variations; Chapter *Density and Stratification*). This approximation states that the **Continuity Equation can be interpreted as div(u) = 0** for waters.

Tip: Familiarize yourself with such situations, by making drawings where the flow field fulfils $\mathrm{div}(\vec{u})=0$. Explain the flow structure in upwelling (equatorial regions of the ocean; coastal regions of lakes) and downwelling zones (mid-latitude central basins in the oceans and lakes).

*c) Momentum equation - Navier-Stokes equation (*Newton's second Law*)*

The **N-S equation** in vector form in a rotating reference frame is

(4.1.5) $$\rho\left(\frac{D\vec{u}}{Dt}+2\vec{\Omega}\times\vec{u}\right)=\rho\,\vec{g}-\mathrm{grad}(p)+\mu\,\nabla^2\vec{u}$$

where $\nabla^2\vec{u}=\mathrm{div}\cdot\mathrm{grad}\,(\vec{u})$ is the **Laplacian of the velocity vector** and describes the diffusion of momentum produced by the viscous force (friction; μ); $\vec{\Omega}$ describes the earth rotation (rotation vector upward; positive = anticlockwise), $\vec{g}$ is the gravity vector (9.81 m s^{-2}), p stands for pressure and μ denotes the dynamic viscosity ($\mu = \rho\cdot\nu$, with ν the kinematic viscosity).

The total derivative $\frac{D\vec{u}}{Dt}$ can be replaced by the partial differentiation as:

(4.1.6) $$\frac{D\vec{u}}{Dt}=\frac{\partial\vec{u}}{\partial t}+\left(\vec{u}\cdot\mathrm{grad}\right)\vec{u}\,.$$

Using the above relation, the N-S equation can be expressed in six terms:

(4.1.7) $$\underset{(I)}{\frac{\partial u_i}{\partial t}}+\underset{(II)}{\sum_j u_j\frac{\partial u_i}{\partial x_j}}+\underset{(III)}{2\left(\vec{\Omega}\times\vec{u}\right)_i}=\underset{(IV)}{-\delta_{i3}g}\;\underset{(V)}{-\frac{1}{\rho}\frac{\partial p}{\partial x_i}}+\underset{(VI)}{\nu\sum_j\frac{\partial^2 u_i}{\partial x_j^2}}$$

- I: local acceleration (storage of momentum; inertia)
- II: change of currents due to local advection (inertial term)
- III: Coriolis acceleration
- IV: gravity, only in the vertical direction (only component 3 in the equation); convention for δ-notation: δ_{13} = δ_{23} = 0, δ_{33} = 1.
- V: acceleration due to pressure-gradient forces
- VI: acceleration due to viscous stress (friction, $\mu / \rho = \nu$ = kinematic viscosity [m^2 s^{-1}])

We can ignore the effect of the Coriolis force ($2\vec{\Omega}\times\vec{u}=0$) for turbulence and small-scale processes. On the contrary, Coriolis forces cannot be ignored for large-scale flows.

The vector cross product can be evaluated as the determinant of a matrix:

$$\Omega\times v=\begin{vmatrix} i & j & k \\ \Omega_x & \Omega_y & \Omega_z \\ v_x & v_y & v_z \end{vmatrix}=\begin{pmatrix} \Omega_y v_z-\Omega_z v_y \\ \Omega_z v_x-\Omega_x v_z \\ \Omega_x v_y-\Omega_y v_x \end{pmatrix},$$

where the vectors ***i***, ***j***, ***k*** are unit vectors in the *x*, *y* and *z* directions. Remember that in index notation the vector cross product is

$$\mathbf{a}\times\mathbf{b}:=\mathbf{c},\quad c_i=\sum_{j,k=1}^{3}\varepsilon_{ijk}a_j b_k$$

where the Levi-Civita symbol in three dimensions has been used:

$$\varepsilon_{ijk}=\begin{cases}+1 & \text{if } (i,j,k) \text{ is } (1,2,3), (3,1,2), \text{or } (2,3,1), \\ -1 & \text{if } (i,j,k) \text{ is } (1,3,2), (3,2,1), \text{or } (2,1,3), \\ 0 & \text{if } i=j \text{ or } j=k \text{ or } k=1 \end{cases}$$

which is 1 if (i, j, k) is an even permutation of (1,2,3), −1 if it is an odd permutation, and 0 if any index is repeated.

Finally, note that the continuity equation for an incompressible flow allows rewriting the advective term in a different form:

(4.1.8) $$\sum_j u_j\frac{\partial u_i}{\partial x_j}=\sum_j\frac{\partial}{\partial x_j}\left(u_i u_j\right)-u_i\sum_j\frac{\partial u_j}{\partial x_j}=\sum_j\frac{\partial}{\partial x_j}\left(u_i u_j\right)$$

which will be more practical for highlighting the turbulent fluxes.

4.2 Reynolds-averaged Navier-Stokes equations (RANS)

a) Reynolds decomposition

The major problem of the application of the Navier-Stokes equation to natural systems is that we cannot possibly measure or calculate all the fluid motions down to the smallest scales. This means that we need some method for simplification. The method of choice is usually to explicitly calculate the average flow field and treat the small scales in a statistical way. The idea is similar to what is generally done to describe molecular diffusivities. Instead of describing the movement of all single molecules, which is simply impossible, the average effect of the ensemble of all movements is described with one single diffusivity coefficient.

As an example, the velocity of a current can be expressed by an average and short-term fluctuations due to turbulence:

(4.2.1) $U_i(t) = u_i(t) + u_i'(t)$ ($u_i = \bar{U}_i$, see definition of averages below)

Note that in this section the instantaneous velocity is indicated with the capital letter U_i, while u_i is the Reynolds-averaged variable.

This procedure of separating the average and the fluctuating part of a quantity is called **Reynolds decomposition**. The Reynolds decomposition is in general not a well-defined procedure. It becomes obvious from *Figure 4.1* that the average flow u = u(t) is a function of time t in general. To find the right time scale τ for forming averages is not at all trivial and to some extent also arbitrary (see example in the exercise on Eddy Correlation).

A classic example is shown in Figure 4.2, where the synoptic component (maximum in spectrum at 4 days) is very well separated from the turbulent fluctuations u' (maximum at about 2 minutes) by a so-called **Energy Gap** (close to ~1 hour) in the **Energy Spectrum**. In this case it appears natural to use a time scale of one hour to calculate the average wind field and define the faster changes as fluctuations. Unfortunately, there is not always such a natural separation between the averages and the fluctuations; especially this gap is often missing in natural waters.

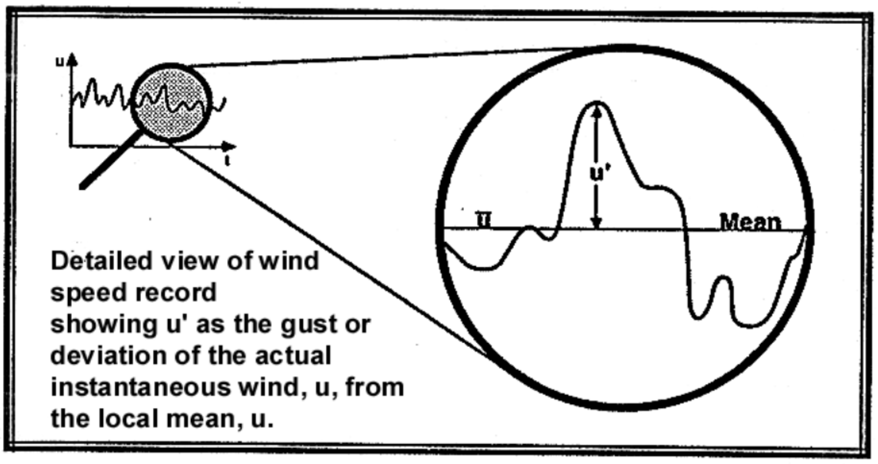


Figure 4.1 – *Schematic wind record with the temporal mean **u** and the wind gust fluctuations **u'**. See also Figure 4.2 for a more detailed explanation concerning the separation of the averages and the fluctuations. From Stull (1988).*

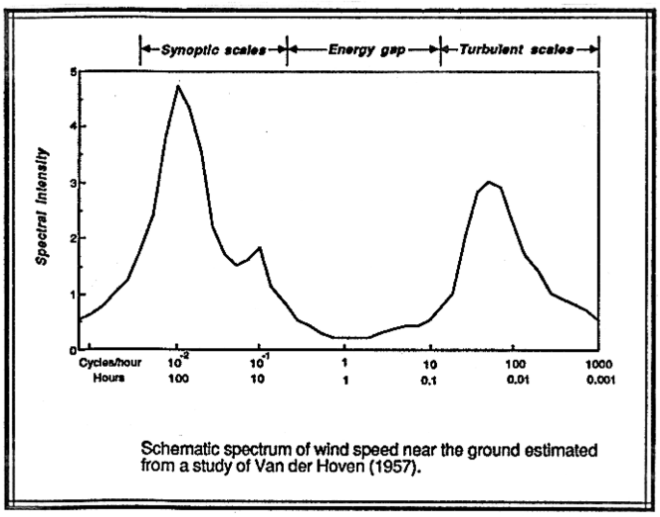


Figure 4.2 – *Spectrum of a wind record close to ground. This is a classical (almost idealized) example, where **u** (the synoptic scale, left peak) is well separated from **u'**, the turbulent fluctuations (right peak).*

There are three options to form averages:

Temporal average $u = \frac{1}{\tau}\int U \, dt$

Spatial average $u = \frac{1}{S}\int U \, ds$ (volume average, areal average, line average; etc.)

Ensemble average $u = \frac{1}{N}\sum_{i=1}^{N} U_i$ (N = number of realizations of identical experiments).

For **homogeneous and stationary** turbulence (in the statistical sense no temporal changes), turbulence fulfils the so called **ergodic conditions**. It says that under those conditions all three averages are identical. The assumption that the temporal statistics is equal to the spatial statistics is often made (Taylor Hypothesis).

b) Navier-Stokes equation with turbulence

Inserting the Reynolds decomposition of the short-term fluctuations into the N-S equation and exploiting eq (4.1.8) leads to:

(4.2.2) $$\frac{\partial}{\partial t}(u_i + u_i') + \sum_j \frac{\partial}{\partial x_j}\left[(u_i + u_i')(u_j + u_j')\right] = -\delta_{i3} g - \frac{1}{\rho}\frac{\partial}{\partial x_i}(p + p') + \nu \sum_j \frac{\partial^2}{\partial x_j^2}(u_i + u_i')$$

Note that in this section we follow the notation of eq (4.2.1): the instantaneous velocity is U_i, while u_i is the Reynolds-averaged variable and u_i' is the fluctuation.

By applying the averaging rules to the equation above, the N-S equation can be simplified as follows:

(4.2.3) $$\underset{(I)}{\frac{\partial u_i}{\partial t}} + \underset{(II)}{\sum_j u_j \frac{\partial u_i}{\partial x_j}} = \underset{(III)}{-\delta_{i3} g} \underset{(IV)}{- \frac{1}{\rho}\frac{\partial p}{\partial x_i}} + \underset{(V)}{\nu \sum_j \frac{\partial^2 u_i}{\partial x_j^2}} \underset{(VI)}{- \sum_j \frac{\partial}{\partial x_j}\left(\overline{u_i' u_j'}\right)}$$

- **I**: local acceleration (storage of **average** momentum; inertia)
- **II:** advection of the average momentum by the **average** flow field
- **III:** gravity
- **IV:** acceleration due to the **average** pressure-gradient forces
- **V:** friction due to the viscous stress of the **average** flow field
- **VI:** **Reynolds stress (covariance of fluctuating velocity components).**

The difference of this temporally averaged N-S equation to the original form above (without fluctuations; N-S equation in laminar form) is only the term VI: **Reynolds stress.** This additional stress is an effect of the non-linear interaction of the velocity components. Therefore, we call it in the following also the **non-linear term** of the N-S Equation. In practice this means that for the description of the **average flow field** the turbulent transport and the turbulent interactions have to be taken into account. The new term - the **Reynolds stress** - acts as an **additional friction** (due to turbulence).

However, in order to deal with this equation, the Reynolds stress term (originated from the non-linearity of the advection term) has to be expressed in terms of averaged variables. The most common closure relationship adopts a Fickian closure (see **Eddy Formulation** in Chapter 3):

(4.2.4) $$\tau_{ij} = -\rho\, \overline{u_i' u_j'} = \rho\, \nu_{ij}^{turb}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

where ν_{ij}^{turb} is called **eddy viscosity**. This implies that the Reynolds stresses can be interpreted as the turbulent diffusion of momentum:

(4.2.5)

$$\underset{(I)}{\frac{\partial u_i}{\partial t}} + \underset{(II)}{\sum_j u_j \frac{\partial u_i}{\partial x_j}} = \underset{(III)}{-\delta_{i3} g} \underset{(IV)}{- \frac{1}{\rho}\frac{\partial p}{\partial x_i}} + \underset{(V)}{\nu \sum_j \frac{\partial^2 u_i}{\partial x_j^2}} + \underset{(VI)}{\sum_j \frac{\partial}{\partial x_j}\left(\nu_{ij}^{turb}\frac{\partial u_i}{\partial x_j}\right)}$$

If an additional assumption that ν^{turb} is homogeneous (i.e. it does not change in space), which is however not true in general, then the Reynolds-average equation becomes formally identical to the Navier-Stokes equation:

(4.2.6)

$$\frac{\partial u_i}{\partial t} + \sum_j u_j \frac{\partial u_i}{\partial x_j} = -\delta_{i3} g - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \left(\nu + \nu^{turb}\right)\sum_j \frac{\partial^2 u_i}{\partial x_j^2}.$$

c) Turbulent kinetic energy - equation for turbulent fluctuations

Let us start again with the original N-S equation for velocity $U_i = u_i + u_i'$, by considering average velocities u_i and fluctuations u_i', the N-S equation for direction i reads:

(4.2.7)

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i'}{\partial t} + \sum u_j \frac{\partial u_i}{\partial x_j} + \sum u_j \frac{\partial u_i'}{\partial x_j} + \sum u_j' \frac{\partial u_i}{\partial x_j} + \sum u_j' \frac{\partial u_i'}{\partial x_j} =$$

$$-\delta_{i3} g - \frac{1}{\rho}\frac{\partial p}{\partial x_i} - \frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2} + \nu \frac{\partial^2 u_i'}{\partial x_j^2}$$

From this total momentum equation, we subtract the time-averaged N-S equation of the average flow field (see above). What remains (after subtraction) is the conservation equation for the turbulent fluctuations u_i':

(4.2.8)

$$\frac{\partial u_i'}{\partial t} + \sum u_j \frac{\partial u_i'}{\partial x_j} + \sum u_j' \frac{\partial u_i}{\partial x_j} + \sum u_j' \frac{\partial u_i'}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \nu \frac{\partial^2 u_j'}{\partial x_j^2} + \frac{\partial}{\partial x_j}\left(\overline{u_i' u_j'}\right)$$

This equation is the starting point for the derivation of the **turbulent kinetic energy (TKE,** $m^2\ s^{-2}$**) equation**. The TKE equation is a fundamental tool linking large scale flow fields and currents u with small-scale turbulence u' and mixing (see below). The TKE is defined as:

(4.2.9) TKE = $0.5 \cdot \overline{(u')^2}$.

From a mathematical perspective, **TKE** is equal to half of the variance $\overline{(u')^2}$. The spectrum of $0.5 \cdot \overline{(u')^2}$ is therefore called the **energy spectrum** (an example is given in *Figure 4.2*). As the TKE is an important link between the large-scale flow (with **kinetic energy KE**, $m^2\ s^{-2}$) and the turbulent mixing (where TKE is relevant), the interpretation of the energy spectrum is important for the understanding of the turbulence dynamics. This is an entire field of science per se.

4.3 Navier-Stokes equation for geophysical flows

Geophysical flows are characterized by the effect of Earth rotation, which - in a rotating reference system - is expressed by the Coriolis acceleration. The **N-S equations in a rotating system** can be written in index notation as:

(4.3.1) $$\frac{Du_i}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} - 2\left(\vec{\Omega} \times \vec{u}\right)_i - \delta_{i3} g + \nu \nabla^2 u_i$$

With $\Omega = (0, \omega \cos(\theta), \omega \sin(\theta))$

- θ : latitude (equator 0°, poles ± 90°)
- ω : angular frequency of Earth rotation = $0.73 \cdot 10^{-4}$ s^{-1} (= 2π / day)

Here the viscous term is written in an approximated way for a homogeneous (no change in space) and isotropic (no change with direction) viscosity, because in general it should be

$$\nu \nabla^2 u_i \leftrightarrow \sum_j \frac{\partial}{\partial x_j}\left(\nu_{ij} \frac{\partial u_i}{\partial x_j}\right)$$

In the following, we will keep the short notation and move to the more complete formulation only when needed. Separating the three components of N-S equations:

(4.3.2) $$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - 2\omega\left(w \cos\theta - v \sin\theta\right) + \nu \nabla^2 u$$

(4.3.3) $$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\omega u \sin\theta + \nu \nabla^2 v$$

(4.3.4) $$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + 2\omega u \cos\theta + \nu \nabla^2 w$$

In the above equations, the meaning of D…/Dt is that of the **total derivative** (eq 4.1.1 above):

(4.3.5) $$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + V\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

The so-called **Euler equations** are the N-S equation above for an ideal fluid with no viscosity ($\nu = 0$). Such an approximation can be made far away from boundaries.

In many geophysical flows the vertical scale is much smaller than the horizontal scales, so the fluid can be thought of as a thin layer on a rotating Earth. In this case the continuity equation implies that

$$W \sim \frac{H}{L} U$$

where *W* is the scale of vertical velocity, *U* is the scale of the horizontal velocity, *H* is a characteristic depth and *L* is the horizontal length scale. If *H/L* is small, *W/U* is small as well and we can introduce some simplifications:

- the "vertical" N-S equation can be approximated in the form of a **hydrostatic pressure distribution**:

(4.3.6) $$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

- we can neglect the effect of the vertical velocity in the Coriolis acceleration and introduce the **Coriolis parameter**

(4.3.7) $$f = 2\omega \sin\theta$$

- the eddy viscosity is not isotropic, meaning that the vertical coefficient ($\nu_{i3} = \nu_z$) is typically different (usually significantly smaller) than the coefficient ($\nu_{i1} = \nu_{i2} = \nu_h$) in the horizontal directions.

As a consequence, the governing equations can be simplified as

(4.3.8) $$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv + \nu_h \nabla_h^2 u + \nu_z \frac{\partial^2 u}{\partial z^2}$$

(4.3.9) $$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu + \nu_h \nabla_h^2 v + \nu_z \frac{\partial^2 v}{\partial z^2}$$

with $\nabla_h^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

The vertical distribution of pressure

(4.3.10) $$\frac{\partial p}{\partial z} = -\rho g \quad \rightarrow \quad p = p_0 - \int_0^z \rho(\zeta)\, g\, d\zeta$$

These three equations above (eqs 4.3.8 – 4.3.10) represent the starting point of most models for the description of flows and circulation in oceans and lakes.

Geophysical large-scale flows

a) Water at rest

If water is at rest, no motions, $u = v = w = 0$ and the equations are simply

(4.4.1) $$\frac{\partial p}{\partial x} = 0\,, \quad \frac{\partial p}{\partial y} = 0\,, \quad \frac{\partial p}{\partial z} = -\rho g$$

This implies that lines of equal pressure (isobars) must be horizontal in the motionless waters. This means that ρ may only be a function of depth *z*, which also includes the condition of a flat (horizontal) surface.

b) Coriolis force - Inertial circulation

The Coriolis force per unit mass $\vec{f}_c$ depends on the cross-product of the earth's angular velocity, $\vec{\Omega}$, which is parallel to its spin axis, and the fluid velocity, **u**, as follows:

(4.4.2) $$\vec{f}_c = -2\vec{\Omega} \times \vec{u}$$ [N kg^{-1} = m s^{-2}]

The Coriolis force is a pseudo force introduced so that we may consider the rotating earth as an inertial system in which Newton's equation holds. Thus the surface velocity, which is primarily due to the wind stress, $\vec{\tau}$, provokes an additional force that is perpendicular to both $\vec{\Omega}$ and $\vec{u}$. The *horizontal* component of $\vec{f}_c$ (active in diverting horizontal flows) varies latitudinal from zero at the equator to a maximum at the poles. At all latitudes, however, this component acts perpendicular to the current (Northern Hemisphere: to the *right of the flow* (fluid to orbit in clockwise circles); Southern Hemisphere: to the *left*). These circular ***inertial circulations*** have a period of the order of one day at mid-latitudes; however, the local ***inertial periods*** range from 12 hours (f = 2 ω sin (90°); τ = 2π/(2ω) = 2π/(4π/day) = 0.5 day) at the poles, to 1 day at 30 ° latitude to **infinity at the equator** (f = 0). These motions are often observed in large lakes following impulsive wind events. See examples in class.

If we consider a flow characterized by an initial value of velocity *U* but far from boundaries, we are allowed to introduce a number of assumptions that make the model much simpler: negligible pressure gradients ($\partial p/\partial x = \partial p/\partial y = 0$), almost frictionless ($\nu_h = \nu_z = 0$), spatially homogeneous ($\partial u/\partial x = \partial u/\partial y = \partial v/\partial x = \partial v/\partial y = 0$). Thus we obtain

(4.4.3) $$\frac{\partial u}{\partial t} = fv$$

(4.4.4) $$\frac{\partial v}{\partial t} = -fu$$

which has the simple solution

(4.4.5) $$u = U\cos(ft), \qquad v = -U\sin(ft)$$

The flow describes a circle rotating clockwise in the northern hemisphere with period $T = 2\pi/f$ (**inertial period**) and radius $R = U/f$ (**inertial radius**). The rotation is counter-clockwise on the southern hemisphere.

Another radius is also used to define the length scale at which rotational effects become as important as buoyancy or gravity wave effects, the **Rossby radius of deformation**

(4.4.6) $$L_R = \frac{\sqrt{g'H}}{f}$$

where *H* is the depth (or layer thickness) and

(4.4.7) $$g' = \frac{\Delta\rho}{\rho} g$$

is the acceleration due to reduced gravity in baroclinic flows. Note that the numerator of L_R is celerity of gravity waves (phase speed) across interfaces between layers with different density.

We can examine whether **neglecting friction terms** is consistent with this result. Typical values for the eddy viscosity in large natural waters are: $\nu_h \approx 10 - 10^5$ m² s⁻¹, $\nu_z \approx 10^{-5} - 10^{-1}$ m² s⁻¹. Firstly we consider horizontal viscosity: since $\nabla_h^2 u \sim \nabla_h^2 v \sim U/R^2$, we can consider the order of magnitude of the friction term as $\nu_h \nabla_h^2 u \sim \nu_h f^2/U$. In order to neglect friction, this has to be smaller (in absolute value) than the Coriolis term $\sim fU$, i.e. $\nu_h << U^2/f$. With *U* ~ 0.1 m s^{-1}, *f* ~ 10^{-4} s^{-1} → U^2/f ~ 10^2 m^2 s^{-1}, so the condition is not always fulfilled. This means that the frictional damping of inertial currents may be significant.

The case of vertical viscosity is more difficult to estimate because the vertical gradient of velocity can be large in the surface mixed layer and depend on the history of the system (since in general, the motions are driven by wind events). The comparison is now between $\nu_z\, \partial^2 u/\partial z^2 \sim \nu_z\, U/H^2$ and the Coriolis term, fU, hence $\nu_z << H^2 f$. For a vertical scale *H* ~ 10^2 m (oceans, deep lakes) this means ν_z << 10^0 m^2 s^{-1}, a condition that is usually fulfilled; on the contrary for *H* ~ 10 m (lakes) it turns out that $\nu_z << 10^{-2}$ m^2 s^{-1}, which is not always fulfilled in the weakly stratified surface layer.

We can also consider the order of magnitude of nonlinear terms: $u\,\partial u/\partial x \sim U^2/R = fU$, which is exactly the same order as the Coriolis term, so the only possibility to neglect them is the assumption of homogeneous flow fields.

In summary: inertial currents feel friction. Their structure ($R = U/f$ ~ 10^3 m) is usually too small to belong to the really large-scale patterns of quasi-frictionless flow. We will return to the effect of friction when dealing with Ekman transport.

c) Wind - the driver of large-scale currents

The wind exerts a shear stress on the water surface, which can be calculated by the empirical relationship:

(4.4.8) $$\vec{\tau} = \rho_a C_{10} W_{10}^2 \qquad [\mathrm{N\ m^{-2}}]$$

where W_{10} is the wind velocity; ρ_a the density of air; and C_{10} the dimensionless aerodynamic drag coefficient (C_{10} stands for wind W_{10} measured 10 m above water). The response of oceans/lakes to the wind stress eq (4.4.8) is not at all in accord with intuition. The intuitive response-motion along the direction of the forcing is significantly altered by **Coriolis force**, **stratification** of the water body, and the redirection of flow due to its continental **boundaries**.

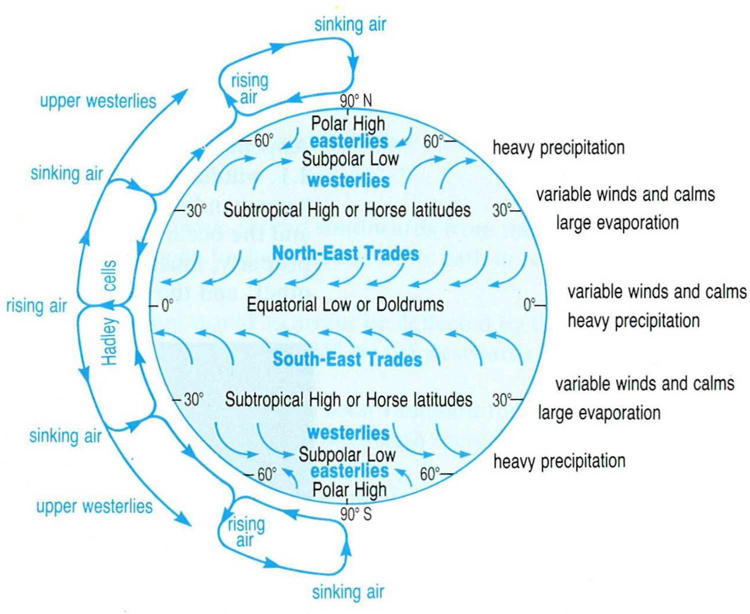


Figure 4.3 – *Global wind patterns. Figure from Open University (1989).*

The above described inertial currents, which are driven by the wind (*Figure 4.3*), are affected by friction. The frictional damping reduces the amplitude of the inertial oscillations and allows the parcel of fluid to take up a component of motion along the direction of the stress, so that the net, long-term surface flow is to the right of the wind stress (Northern Hemisphere) at some angle, typically 10° to 45°. Such flow is termed **Ekman wind drift**. The near-surface Ekman flow in turn exerts a frictional stress on the layer of fluid immediately below, which also responds by moving to the right of the surface flow, but at a somewhat reduced velocity because of friction. This veering effect continues to migrate downward, so that local velocity vector of the oceanic current rotates continuously to the right with depth, while decaying exponentially with a scale of order 10 to 20 m (*Figure 4.4*).

This volume flow per unit horizontal distance is termed **Ekman transport** (units $\mathbf{m^2\ s^{-1}}$). On the equatorward flanks of the easterly **trade winds**, the right-angle forcing moves surface water poleward from the equatorial region and results in cold, subsurface water flowing upward to replace the missing surface water, a process called **equatorial upwelling** (*Figure 4.5*).

Ekman transport contributes to the first step in the formation of the major subtropical and subpolar **oceanic gyres** as well as the **equatorial current systems**. In a typical ocean basin (Figure 4.6a) both the easterly trade winds and the mid-latitude westerlies force surface water toward the gyre interior because of Ekman transport (Figure 4.6b). The **surface convergence** causes an accumulation of warmer, lighter water, and results in a small elevation (order ~1 m or less) of the surface above the equipotential. It also causes a much larger **deepening of the thermocline**, which is found down to depths of several hundred meters. Thus, Ekman inflow into the subtropical convergence (interior of the gyre) results in a ***downwelling*** of the surface waters (Figure 4.6b). The mathematical derivation of the Ekman phenomenon and the Ekman transport are detailed in the following Chapter 4.5.

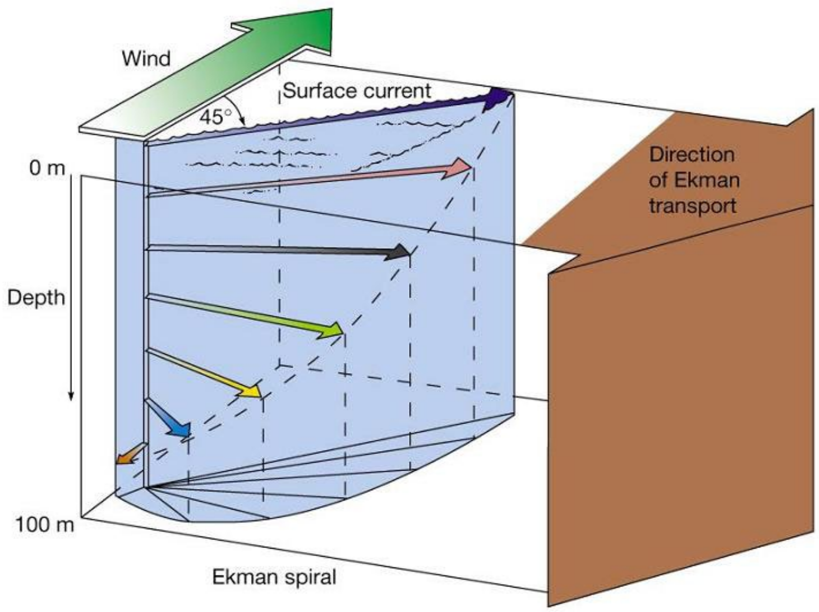


Figure 4.4 *– Rotation and attenuation of the near-surface velocity vector with depth through the surface Ekman layer of a water body. Wind direction is indicated by topmost vane. If the wind drift current is integrated over depths, the net transport of water is at 90° to the direction of the wind stress (to right on the Northern Hemisphere; to left in Southern Hemisphere) (Figure: San Francisco State University).*

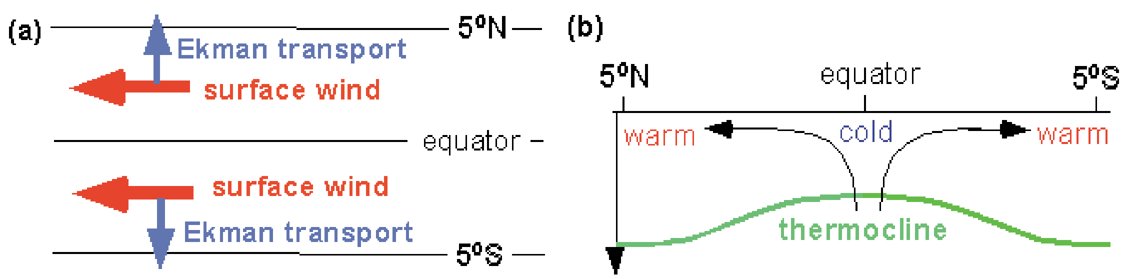


Figure 4.5 – ***Ekman sucking*** *along the equator. (a) Plan view of the prevailing surface wind and resulting water transport away from the equator in the* ***Ekman layers*** *of the oceans. Therefore, the equator is a* ***divergence zone****. (b) Corresponding cross section, showing the* ***upwelling*** *and resulting SST anomalies which is a consequence of the* ***equatorial divergence*** *[University of Wyoming].*

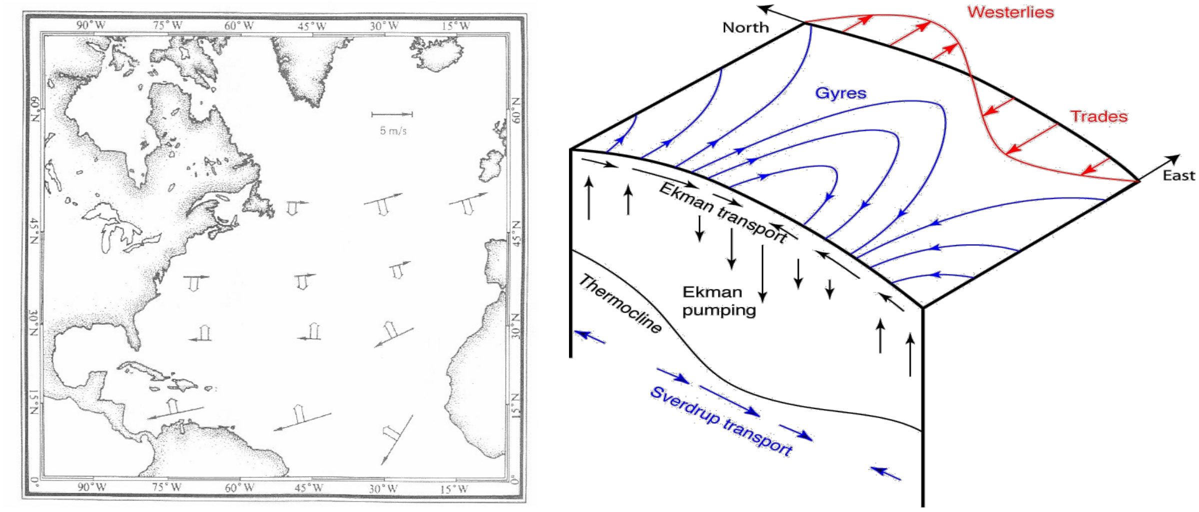


Figure 4.6 – *(left)* ***Convergence*** *of surface water toward the interior of the* ***North Atlantic Gyre*** *under the influence of* ***Ekman transport****. Both* ***trade winds*** *(at equator) and* ***westerlies*** *(mid-northern latitude) contribute to the accumulation of the warm water pool in the center of the* ***North Atlantic Gyre****, creating a* ***convergence zone*** *which leads to* ***downwelling****. Figure from Apel [1987].*
(right) Wind-driven general circulation, including subduction as well as upwelling [University of California, San Diego].

d) Global geostrophic flows in oceans

The small surface elevation indicated in Figure 4.6b is termed ***setup****,* and has associated with it a hydraulic head and a horizontal pressure gradient, $-\nabla_h p$, that is opposed by the horizontal component of Coriolis force, $\rho\left(2\vec{\Omega}\times\vec{u}\right)_h$. The currents resulting from this balance of forces are called ***geostrophic* flows** (earth-turning), and in the absence of time variations and friction, move approximately along surfaces of constant elevation. Thus the geostrophic equation is:

(4.4.9) $$\nabla_h p = -\rho\left(2\vec{\Omega}\times\vec{u}\right)_h \qquad [\mathrm{N\ m^{-3}}]$$

where h denotes the horizontal component. Geostrophic balance is not confined to surface currents, but may exist at all depths. This equation for balanced flow forms an important tool in the analysis of currents in large natural waters. The two horizontal components for the geostrophic currents for steady flows (index g refers to geostrophic) read:

(4.4.10) $$fv = \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \text{or} \qquad v_g = \frac{1}{\rho f}\frac{\partial p}{\partial x}$$

(4.4.11) $$fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad \text{or} \qquad u_g = -\frac{1}{\rho f}\frac{\partial p}{\partial y}$$

(4.4.12) or in summary: $$\vec{u}_g = \frac{1}{\rho f}\left(\vec{e}_z \times \nabla_h p\right)$$

Geostrophic currents are steady flows in which the horizontal pressure gradients balance Coriolis acceleration over long lateral distances (100s to 1000s of km). The vertical pressure gradient is in balance with gravity. The flows are perpendicular to the horizontal pressure gradient and parallel to the isobars (lines of constant pressure). Thus, in the northern Hemisphere the high pressure region is on the right of the flow. This means that **nonlinear acceleration and turbulent friction are neglected** in comparison to Coriolis force. In particular, if *U* is the scale for velocity and *L* for length (i.e., distance over which horizontal gradients of the variable occur), we can compare advective acceleration $\sim U^2/L$ and Coriolis acceleration $\sim fU$. The ratio is known as **Rossby number**:

(4.4.13)
$$Ro = \frac{nonlinear\ \ acc.}{Coriolis\ \ acc.} = \frac{U}{fL}$$

The geostrophic approximation is valid for small values of the Rossby number, a condition that is more common in oceans than in lakes. It is actually a standard approximation in atmospheric studies at the synoptic scale (U ~ 10 m s^{-1}, f ~ 10^{-4} s^{-1}, L ~ 1000 km → Ro ~ 1).

Note: While the horizontal pressure gradients are balanced by Coriolis force, the vertical pressure gradient is compensated by the gravitational acceleration in the hydrostatic approximation.

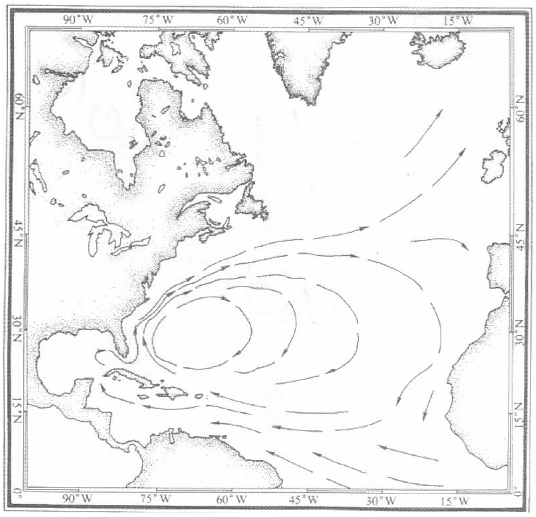


Figure 4.7
Surface streamlines of an idealized subtropical gyre. Intensification of flow along the western boundary is due to the overall vorticity balance of the gyre. Figure from Apel [1987].

Applied to Figure 4.6, this relation implies that at the equator the flow is to the west (from Africa to South America; **North Equatorial Current**), whereas the return flow at mid-latitudes (region of westerlies) is directed to the east (North America to Europe; **North Atlantic Drift;** Figure 4.7). The compensating south-north flows are concentrated to the boundary currents – stronger on the western side (Eastern USA; China) for secondary hydrodynamic reasons (not discussed here) and weaker on the eastern side. The south-to-north flow is the **Florida Current / Gulfstream** on the western side of the basin (**Kuroshio Current** in the Pacific). The north-to-south flow is the **Canary Current** on the eastern side of the basin (Portugal, Africa) which reaches as far as Senegal before it turns west and joins the North Equatorial Current (Figure 4.7). The center of the **North Atlantic Gyre** is an area of high pressure (**anticyclones**) and the gyre turns clockwise. Opposite: **cyclones** (low pressure area) rotate anticlockwise. Southern Hemisphere: since $f < 0$, the flow direction is reversed (greater pressure to the left).

The geostrophic flows are perpendicular to the horizontal pressure gradient $\nabla_h p$ and parallel to the isobars (lines of constant pressure). Thus, in the northern hemisphere the high pressure region is on the right of the flow. This also means that cyclones (area of low pressure) rotate anticlockwise and anticyclones (areas of high pressure) rotate clockwise. Vice versa, since $f < 0$ in the southern hemisphere, the flow direction is reversed (greater pressure to the left).

Formally, the geostrophic flow can also be represented by the vector product

(4.4.14) $$\vec{u}_g = \frac{1}{\rho f}\left(\vec{e}_z \times \nabla_h p\right)$$

Where $\vec{e}_z = (0,0,1)$ is the unit vector in the z-direction.

(4.5.5) $$\frac{\partial U}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fV + \nu_z \frac{\partial^2 U}{\partial z^2}$$

(4.5.6) $$\frac{\partial V}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fU + \nu_z \frac{\partial^2 V}{\partial z^2}$$

For steady-state conditions ($\partial U/\partial t = \partial V/\partial t = 0$), they become even simpler:

(4.5.7) $$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fV + \nu_z \frac{\partial^2 U}{\partial z^2}$$

(4.5.8) $$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fU + \nu_z \frac{\partial^2 V}{\partial z^2}$$

and we obtain the so-called Ekman flow. To solve this system of equations we need two boundary conditions (e.g. at the surface *z* = 0 and at the bottom *z* = -*h*) for each of the two velocity components.

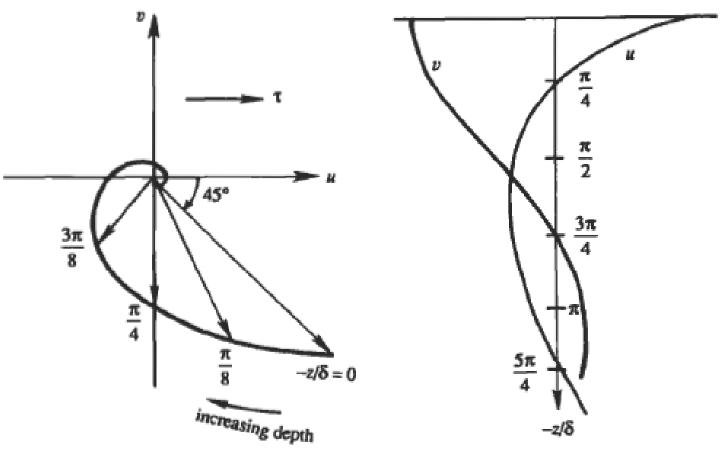
Two dimensionless parameters can be introduced to describe the ratio between the friction term and the Coriolis term (definitions with factor of 2 are also used; not relevant):

(4.5.9) $$E_h = \frac{horizontal \quad friction}{Coriolis \quad acc} = \frac{\nu_h}{fL^2}$$ horizontal Ekman number

(4.5.10) $$E_z = \frac{vertical \quad friction}{Coriolis \quad acc} = \frac{\nu_z}{fH^2}$$ vertical Ekman number.

b) Ekman transport at the surface without pressure gradients

In this section, we chose as symbols for horizontal currents (U, V), simply in order not to confuse with the symbol ν, which stand for viscosity (in the equation editor minor v reads as ν which is very similar to ν). For sake of clarity, we first analyse the simplest case, introducing some assumptions: the depth is very large so we can assume that the bottom boundary condition is at *z* = -∞; neglect the pressure gradients (that are related to geostrophic flow).



(a) Hodograph (b) Profiles of *u* and *v*

Figure 4.17 – *Ekman layer at a free surface: (a) the velocity vector at different heights (δ in the figure is D_E); (b) the velocity profiles [Kundu and Cohen, 2002].*

The governing equations are

(4.5.11) $$\frac{\partial^2 U}{\partial z^2} = -\frac{f}{\nu_z} V$$

(4.5.12) $$\frac{\partial^2 V}{\partial z^2} = \frac{f}{\nu_z} U$$

with the boundary conditions (for a wind directed northward):

(4.5.13) $$\left.\frac{\partial U}{\partial z}\right|_{z=0} = 0\,, \quad \left.\frac{\partial V}{\partial z}\right|_{z=0} = \frac{\tau}{\rho \nu_z}\,, \quad U\big|_{z=-\infty} = 0\,, \quad V\big|_{z=-\infty} = 0$$

By differentiation the first equation twice in respect to z and substituting into the second one, a fourth-order ordinary differential equation is obtained:

(4.5.14) $$\frac{\partial^4 U}{\partial z^4} = -\left(\frac{f}{\nu_z}\right)^2 U$$

An identical equation can be derived for *V*. Using the boundary conditions, the solution is

(4.5.15) $$U = V_0 \exp\left(\frac{z}{D_E}\right)\cos\left(\frac{\pi}{4} + \frac{z}{D_E}\right)$$

(4.5.16) $$V = V_0 \exp\left(\frac{z}{D_E}\right)\sin\left(\frac{\pi}{4} + \frac{z}{D_E}\right)$$ where

(4.5.17) $$V_0 = \frac{\tau}{\rho\sqrt{\nu_z f}}$$

is the scale of velocity, and

(4.5.18) $$D_E = \sqrt{\frac{2\nu_z}{f}}$$

is the depth of the so-called Ekman layer.

The flow field described by the solution is a spiral for the velocity vector, which changes direction while the velocity is decreasing in downward direction. In fact, the winds at the surface move the water on the surface. This movement is transferred to the underlying layers through turbulent (eddy) viscosity (friction). However, the Coriolis force tends to deviate the water movement (to the right on the northern hemisphere) causing a direction of the velocity at the surface that differs 45° from the wind direction. Apparently unexpected, the net water transport integrated over the entire depth is orthogonal to the wind. We can see these features analytically observing that the velocity at the surface is

(4.5.19) $$U = V_0 \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} V_0\,, \quad V = V_0 \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} V_0$$

so that the module of the velocity is exactly $|\vec{U}| = \sqrt{U^2 + V^2} = V_0$.

The total transport (per unit length, with the units [$m^2\ s^{-1}$]) produced by such a flow can be calculated as

(4.5.20) $$q_x = \int_{-\infty}^{0} U\,dz = \frac{V_0 D_E}{\sqrt{2}} = \frac{\tau}{\rho f}$$

(4.5.21) $$q_y = \int_{-\infty}^{0} V\,dz = 0$$

confirming that, on average, no transport occurs along the direction of the wind, and that a net transport exists at 90° (rightward in the northern hemisphere, leftward in the southern).

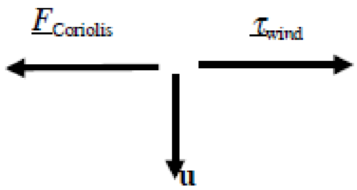


Figure 4.18 – *Balance of forces within the Ekman layer.*

It is also possible to rewrite the governing equations as

(4.5.22) $$\frac{\partial \tau_x}{\partial z} = -\rho f V \qquad \tau_x = \rho\, \nu_z \frac{\partial U}{\partial z} ,$$

(4.5.23) $$\frac{\partial \tau_y}{\partial z} = \rho f U \qquad \tau_y = \rho \nu_z \frac{\partial V}{\partial z} .$$

These equations, integrated from the top (where $\vec{\tau}$ is equal to the wind stress) to infinity (where $\vec{\tau}$ goes to zero), provides directly the solution eqs (4.5.19 and 4.5.20). This result is more general than the Ekman spiral and suggests that the balance of the forces within the Ekman layer is between the wind shear stress and the Coriolis force (Figure 4.18).

References

1. Kundu P.K. and I.M. Cohen (2002), Fluid mechanics, 2nd ed., Academic Press, Elsevier.
2. Schwab, D. J. and D. Beletsky (2003). Relative effects of wind stress curl, topography, and stratification on large-scale circulation in Lake Michigan. Journal of Geophysical Research 108(C2): 26-1 - 26-10.